The symmetry that does all the work

Let

$$S(x) = \left(1 + x + x^2\right)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}.$$

Because the highest power is $2n$, multiply $S(1/x)$ by x^{2n}:

$$x^{2n} S\left(\frac{1}{x}\right) = x^{2n}\left(1 + \tfrac{1}{x} + \tfrac{1}{x^2}\right)^n = x^{2n}\, x^{-2n}\left(1 + x + x^2\right)^n = S(x).$$

But

$$x^{2n} S\left(\frac{1}{x}\right) = x^{2n} \sum_{k=0}^{2n} a_k x^{-k} = \sum_{k=0}^{2n} a_k x^{2n-k} = \sum_{k=0}^{2n} a_{2n-k} x^k.$$

Comparing the two series term-by-term (uniqueness of power-series coefficients) gives the *palindromic* relation

$$\boxed{a_k = a_{2n-k} \qquad (0 \le k \le 2n).}$$

Apply it to the required indices

Set $k = n - 1$:

$$a_{n-1} = a_{2n-(n-1)} = a_{n+1}.$$

Hence the coefficients of x^{n-1} and x^{n+1} in $(1 + x + x^2)^n$ are indeed equal.

Why this symmetry is natural (optional intuition)

Algebraic view The polynomial $1 + x + x^2$ is **self-reciprocal**:
$x^2\left(1 + 1/x + 1/x^2\right) = 1 + x + x^2$.
Raising to the n^{th} power preserves that property, so the whole expansion is palindromic.

Combinatorial view Each term of coefficient a_k corresponds to a way of picking n numbers, each $0, 1, 2$, whose sum is k.
If you swap every 0 with a 2 in such a selection, its total changes from k to $2n - k$.
That establishes a bijection between the sets counted by a_k and a_{2n-k}, so the two coefficients are equal.